Joint probability distributions for a class of non-Markovian processes

A. Baule and R. Friedrich
Institute of Theoretical Physics, Westfälische Wilhelms-Universität Münster, Wilhelm-Klemm-Strasse 9, G-48149 Münster, Germany
(Received 4 August 2004; published 1 February 2005)

We consider joint probability distributions for the class of coupled Langevin equations introduced by Fogedby [H. C. Fogedby, Phys. Rev. E **50**, 1657 (1994)]. We generalize well-known results for the single-time probability distributions to the case of N-time joint probability distributions. It is shown that these probability distribution functions can be obtained by an integral transform from distributions of a Markovian process. The integral kernel obeys a partial differential equation with fractional time derivatives reflecting the non-Markovian character of the process.

DOI: 10.1103/PhysRevE.71.026101 PACS number(s): 02.50.−r, 05.40.−a, 47.27.−i, 05.30.Pr

I. INTRODUCTION

In recent years, the connections between the continuous time random walk (CTRW), which originated in the work of Montroll and Weiss [1], generalizing the idea of Brownian random walks, and fractional Fokker-Planck equations have been established. For a review we refer the reader to [2]. The solutions of these equations exhibit both super- and subdiffusive behavior and are thus appropriate models for a large variety of transport processes in complex systems [3]. Recently, a connection between the velocity increment statistics of a Lagrangian tracer particle in fully developed turbulent flows and a type of CTRW has been introduced [4]. Here, a closure assumption on a hierarchy of joint velocity-position probability distribution functions (PDF's) derived from a statistical formulation of the Navier-Stokes equation leads to a generalization of Obukhov's random walk model [5] in terms of a continous time random walk. It allows for a successful parametrization of the single-time probability distributions of velocity increments. However, there are different suggestions for the stochastic process of Lagrangian particles in turbulence, which are able to provide reasonable approximations for the single-time velocity increment statistics. This example evidences that one has to introduce further quantities in order to distinguish between different stochastic models.

For non-Markovian processes, the natural extension is the consideration of N-time joint probability distributions. It seems that for the class of CTRW's only single-time probability distributions have been investigated so far. In that case fractional diffusion equations of the form

$$\frac{\partial}{\partial t}f(x,t) = {}_0D_t^{1-\alpha}Lf(x,t) \tag{1}$$

can be derived. Here x denotes the random variable, L is a Fokker-Planck operator (for diffusion processes $L=\partial^2/\partial x^2$), and ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional differential operator (see Appendix A). The properties of this equation with regard to physical applications have been extensively discussed in the recent reviews [2,6]. In [7] Fogedby introduced a class of coupled Langevin equations, where he also considered a case which leads to an operator L including fractional derivatives with respect to the variable x, $L=\partial^\beta/\partial x^\beta$. A similar case has been studied by Meerschaert *et al.* [8], who made an extension to several dimensions introducing a multidimensional generalization of fractional diffusion, so-called operator Lévy motion. This allows for a description of anomalous diffusion with direction-dependent Hurst indices H_i defined by the relation $\langle[x_i(t)-x_i(t=0)]^2\rangle \approx t^{2H_i}$. In [9] limit theorems of a class of continuous time random walks with infinite mean waiting times have been investigated. It is shown that the limit process obeys a fractional Cauchy problem. The emphasis again is put on single-time distributions.

The purpose of the present paper is to investigate multiple-time probability distribution functions for the class of coupled Langevin equations introduced by Fogedby [7], which have been considered to be a representation of a continuous time random walk.

The paper is outlined as follows. In the next section we present the coupled Langevin equations considered by Fogedby [7] consisting of a usual Langevin process $X(s)$ in a coordinate s and a Lévy process representing a stochastic relation $t(s)$. One is interested in the process $X(t)=X(s^{-1}(t))$. Fogedby [7] investigated the case where the processes $X(s)$ and $t(s)$ are statistically independent and showed how fractional diffusion equations of the form (1) arise. Interesting results for the case where the processes are statistically dependent have been considered by Becker-Kern *et al.* [10] leading to generalizations of the fractional diffusion equations (1). However, both publications are devoted to single-time probability distributions.

In Sec. II we present a central formula, which relates the N-time probability distributions of $X(t)$ to the PDF's of $X(s)$ via an integral transform, which is determined by the process $t(s)$. In Sec. III properties of the involved Lévy-stable process $s(t)$ are considered, leading to expressions for the PDF of the inverse process $s(t)$. In Sec. V we specify the moments for the case of a simple diffusion process.

II. A CLASS OF NON-MARKOVIAN PROCESSES

The starting point of our discussion is the set of coupled Langevin equations [7] for the motion of a Brownian particle in an external force field F in $d=1$ dimensions (an extension to higher dimensions $d>1$ is straightforward):

$$\frac{dX(s)}{ds} = F(X) + \eta(s), \tag{2}$$

$$\frac{dt(s)}{ds} = \tau(s). \tag{3}$$

In this framework the random walk is parametrized in terms of the continuous path variable s, which may be considered, e.g., as the arclength along the trajectory. $X(s)$ and $t(s)$ denote the position and time in physical space. The random variables $\eta(s)$ and $\tau(s)$ are responsible for the stochastic character of the process. We are only considering the case of uncoupled jump lengths and waiting times such that η and τ are statistically independent (coupled CTRW's have been considered in [10]). The arclength is related to physical time t by the inverse function $s=t^{-1}(t)=s(t)$. Thus, we have to assume $\tau(s)>0$. We are interested in the process $X(s(t))$, i.e., the behavior of the variable X as a function of physical time t.

For the characterization of the process we introduce the two-time probability density functions for the processes (2) and (3):

$$f_1(x_2,s_2;x_1,s_1) = \langle \delta(x_2 - X(s_2))\,\delta(x_1 - X(s_1))\rangle, \tag{4}$$

$$p(t_2,s_2;t_1,s_1) = \langle \delta(t_2 - t(s_2))\,\delta(t_1 - t(s_1))\rangle, \tag{5}$$

$$f(x_2,t_2;x_1,t_1) = \langle \delta(x_2 - X(s(t_2)))\,\delta(x_1 - X(s(t_1)))\rangle. \tag{6}$$

Here the brackets $\langle\cdots\rangle$ denote a suitable average over stochastic realizations. For the sake of simplicity we restrict ourselves to $n=2$. The generalization to multiple times is obvious. Both probability functions are determined by the statistics of the independent random variables η and τ.

A. The process $X(s)$

We consider the case where $\eta(s)$ is the standard Langevin force, i.e., η is a Wiener process. In turn Eq. (2) becomes Markovian and $f_1(x_2,s_2;x_1,s_1)$ can be determined by solving the corresponding Fokker-Planck equation (FPE) for the conditional probability distribution $P(x_2,s_2|x_1,s_1)$:

$$\frac{\partial}{\partial s}P(x_2,s_2|x_1,s_1) = \left(-\frac{\partial}{\partial x}F(x) + \frac{\partial^2}{\partial x^2}\right)P(x_2,s_2|x_1,s_1)$$
$$= L_{FP}(x)P(x_2,s_2|x_1,s_1). \tag{7}$$

The diffusion constant is set to 1 in the following. Due to the Markovian property of the process $X(s)$ the joint PDF is obtained by multiplication with the single-time PDF according to

$$f_1(x_2,s_2;x_1,s_1) = P(x_2,s_2|x_1,s_1)f(x_1,s_1). \tag{8}$$

For a general treatment of the FPE we refer the reader to the monographs of Risken [11] and Gardiner [12].

B. The process $t(s)$

The stochastic process $t(s)$ is determined by the properties of $\tau(s)$. The corresponding PDF's are denoted by $p(t,s)$ and $p(t_2,s_2;t_1,s_1)$. Furthermore, we shall consider $\tau(s)$ to be a (one-sided) Lévy-stable process of order α [7,13] with $0<\alpha<1$. As a result, the process $t(s)$ is Markovian. Lévy-stable processes of this kind induce the property of a diverging characteristic waiting time $\langle t(s)\rangle$. Consequently the stochastic process in physical time t, given by the coupling of the Langevin equations (2) and (3) reveals subdiffusive behavior. The specific form of $p(t_2,s_2;t_1,s_1)$ will be given below.

For a deeper discussion we refer to the review articles [2,4,6] where the general relation between subdiffusive behavior and diverging waiting times has been treated in detail.

C. The process $X(t)=X(s(t))$

We are interested in the properties of the variable X with respect to the physical time t. Therefore, we have to consider the inverse of the stochastic process $t=t(s)$:

$$s = t^{-1}(t) = s(t). \tag{9}$$

The stochastic process $X(s(t))$ then is described by the joint probability distribution

$$f(x_2,t_2;x_1,t_1) = \langle \delta(x_2 - X(s_2))\,\delta(s_2 - s(t_2))\,\delta(x_1 - X(s_1))\,\delta(s_1 - s(t_1))\rangle. \tag{10}$$

The N-point distributions are determined in a similar way. Introducing the probability distribution h for the inverse process $s(t)$,

$$h(s,t) = \langle \delta(s - s(t))\rangle,$$

$$h(s_2,t_2;s_1,t_1) = \langle \delta(s_2 - s(t_2))\,\delta(s_1 - s(t_1))\rangle, \tag{11}$$

we can calculate the PDF of the process $X(t)=X(s(t))$ as a function of the physical time by eliminating the path variables s_i:

$$f(x_2,t_2;x_1,t_1) = \int_0^\infty ds_1 \int_0^\infty ds_2 h(s_2,t_2;s_1,t_1) f_1(x_2,s_2;x_1,s_1). \tag{12}$$

This relationship is due to the fact that the processes $X(s)$ and $t(s)$ are statistically independent. In that case, the expectation values in Eq. (10) factorize. Equation (12) can be generalized to N times. In fact, one may turn over to a path integral representation:

$$f(x(t)) = \int \mathcal{D}s(t) h(s(t)) f_1(x(s(t))). \tag{13}$$

However, we do not investigate this path integral further.

The probability distribution h can be determined with the help of the cumulative distribution function of $s(t)$. Since the process $t(s)$ has the property (for $s>0$) $s_2>s_1 \rightarrow t(s_2)>t(s_1)$, one has the relationship

$$\Theta(s - s(t)) = 1 - \Theta(t - t(s)). \tag{14}$$

Here, we have introduced the Heaviside step function $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$, $\Theta(x=0)=1/2$. The validity of Eq. (14) becomes evident from an inspection of Fig. 1: The function $\Theta(s-s(t))$ equals 1 in the region above the

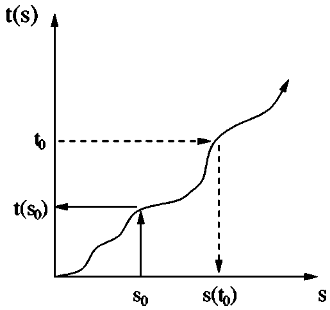

FIG. 1. Sketch of the process $t(s)$ which relates the arclength s to physical time t. Since the increment $\tau(s)$ of Eq. (3) is positive, the curve $t(s)$ is monotonically increasing, implying the validity of the relation (14).

curve $t=t(s)$, whereas $\Theta(t-t(s))$ equals 1 in the region below the curve $t=t(s)$. On the curve $\Theta(s-s(t))=1/2=\Theta(t-t(s))$.

An immediate consequence is the following connection among the cumulative distribution functions of the processes $t(s)$ and $s(t)$:

$$\langle\Theta(s-s(t))\rangle = 1-\langle\Theta(t-t(s))\rangle,$$

$$\begin{aligned}\langle\Theta(s_2-s(t_2))\Theta(s_1-s(t_1))\rangle &= \langle[1-\Theta(t_2-t(s_2))][1-\Theta(t_1-t(s_1))]\rangle\\ &= 1-\langle\Theta(t_2-t(s_2))\rangle-\langle\Theta(t_1-t(s_1))\rangle\\ &\quad+\langle\Theta(t_2-t(s_2))\Theta(t_1-t(s_1))\rangle.\end{aligned} \quad (15)$$

Simple differentiation of Eq. (15) yields the probability density function h of the process $s(t)$:

$$h(s,t) = -\frac{\partial}{\partial s}\langle\Theta(t-t(s))\rangle,$$

$$h(s_2,t_2;s_1,t_1) = \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\langle\Theta(t_2-t(s_2))\Theta(t_1-t(s_1))\rangle. \quad (16)$$

Furthermore, since for $t=0$ we have the correspondence $s=0$, the usual boundary conditions hold:

$$h(s,0) = \delta(s),$$

$$h(s_2,t_2;s_1,0) = h(s_2,t_2)\delta(s_1),$$

$$h(s_2,t_2\to t_1;s_1,t_1) = \delta(s_2-s_1)h(s_1,t_1), \quad (17)$$

and can be verified from Eq. (16).

III. DETERMINATION OF THE PROBABILITY DISTRIBUTIONS $p(s,t)$: LÉVY-STABLE PROCESSES

In the following we shall consider the joint multiple-time PDF of the Lévy-stable process (3) of order α. Simple integration of Eq. (3) yields

$$t(s_i) = \int_0^{s_i} ds'\,\tau(s'), \quad (18)$$

where we assume $\tau(s)>0$. Additionally, we consider the characteristic function for $\omega=i\lambda$. This defines the Laplace transform

$$\begin{aligned}Z(\lambda_2,s_2;\lambda_1,s_1) &:= \mathcal{L}\{p(t_2,s_2;t_1,s_1)\}\\ &= \int_0^\infty dt_2\int_0^\infty dt_1 e^{-\lambda_2 t_2-\lambda_1 t_1}p(t_2,s_2;t_1,s_1).\end{aligned} \quad (19)$$

It will become clear below that working with Laplace transforms is more convenient for manipulating the PDF's of process (3) in the present context.

A. One-sided Lévy-stable processes: Single time

At this point we have to introduce specific properties of the Lévy-stable process. Lévy distributions $L_{\alpha,\beta}(x)$ are defined by two parameters [14,15]: α characterizes the asymptotic behavior of the stable distribution for large x and hence the critical order of diverging moments. β characterizes the asymmetry. In the present case $\tau>0$ and the distribution is maximally asymmetric $p(t<0,s)=0$. This leads to $\beta=1$. In the following we denote the Lévy distribution $L_{\alpha,\beta}(x)$ for $\beta=1$ by $L_\alpha(x)$.

Let us motivate the consideration of Lévy statistics. To this end we consider the characteristic function, which we write in the form

$$Z(\lambda,s) = \left\langle \exp\left(-\lambda s^{1/\alpha}\frac{1}{s^{1/\alpha}}\int_0^s ds'\,\tau(s')\right)\right\rangle, \quad (20)$$

where α is a certain parameter. The choice $Z(\lambda,s)=\tilde{Z}(\lambda^\alpha s)$ leads to a scale invariant PDF $p(t,s)=1/s^{1/\alpha}P(t/s^{1/\alpha})$ [8].

As a result, the characteristic function takes the form

$$Z(\lambda,s) = e^{-\lambda^\alpha s}, \quad (21)$$

where we assume $0<\alpha<1$.

The probability distribution then becomes

$$p(t,s) = \frac{1}{s^{1/\alpha}}L_\alpha\left(\frac{t}{s^{1/\alpha}}\right), \quad (22)$$

where $L_\alpha(t)$ denotes the one-sided Lévy-stable distribution whose Laplace transform is $\mathcal{L}\{L_\alpha(t)\}=e^{-\lambda^\alpha}$.

B. Multiple times

The joint PDF of the Lévy process $t(s)$ has been introduced in Eq. (5). Starting with this definition the derivation of the explicit expression for the PDF is straightforward and clearly reveals the Markovian character of this process. The characteristic function is given as Laplace transform of Eq. (5):

$$Z(\lambda_2,s_2;\lambda_1,s_1) = \int_0^\infty dt_2 \int_0^\infty dt_1 e^{-\lambda_2 t_2 - \lambda_1 t_1} p(t_2,s_2;t_1,s_1) = \left\langle \exp\left(-\lambda_2 \int_0^{s_2} ds'\,\tau(s') - \lambda_1 \int_0^{s_1} ds'\,\tau(s') \right) \right\rangle . \tag{23}$$

For further evaluating this expression we have to distinguish between the cases $s_2 > s_1$ and $s_1 > s_2$. With a given ordering of s_2, s_1 we can rearrange the integrals and write Z as a sum of two contributions:

$$Z(\lambda_2,s_2;\lambda_1,s_1) = \Theta(s_2 - s_1)\left\langle \exp\left(-\lambda_2 \int_{s_1}^{s_2} ds'\,\tau(s') - (\lambda_1+\lambda_2)\int_0^{s_1} ds'\,\tau(s') \right) \right\rangle + \Theta(s_1 - s_2) \times \left\langle \exp\left(-\lambda_1 \int_{s_2}^{s_2} ds'\,\tau(s') - (\lambda_1+\lambda_2)\int_0^{s_2} ds'\,\tau(s') \right) \right\rangle . \tag{24}$$

Here the expectation values factorize due to statistical independence of the increments τ and can be expressed according to Eq. (21):

$$Z(\lambda_2,s_2;\lambda_1,s_1) = \Theta(s_2 - s_1) e^{-s_1(\lambda_1+\lambda_2)^\alpha} e^{-(s_2-s_1)\lambda_2^\alpha} + \Theta(s_1 - s_2) e^{-s_2(\lambda_1+\lambda_2)^\alpha} e^{-(s_1-s_2)\lambda_1^\alpha} . \tag{25}$$

This is the characteristic function of the Lévy process for multiple times. The appearance of the exponents $(\lambda_1+\lambda_2)^\alpha$ is characteristic in this context and carries over to the PDF of the inverse process. We obtain the PDF $p(s_2,t_2;s_1,t_1)$ after performing the inverse Laplace transform of Eq. (25). The result is

$$p(t_2,s_2;t_1,s_1) = \Theta(s_2 - s_1)\frac{1}{(s_2-s_1)^{1/\alpha}} L_\alpha\left(\frac{t_2-t_1}{(s_2-s_1)^{1/\alpha}}\right)\frac{1}{s_1^{1/\alpha}} \times L_\alpha\left(\frac{t_1}{s_1^{1/\alpha}}\right) + \Theta(s_1 - s_2)\frac{1}{(s_1-s_2)^{1/\alpha}} \times L_\alpha\left(\frac{t_1-t_2}{(s_1-s_2)^{1/\alpha}}\right)\frac{1}{s_2^{1/\alpha}} L_\alpha\left(\frac{t_2}{s_2^{1/\alpha}}\right). \tag{26}$$

This expression explicitly exhibits the Markovian nature of the process. The conditional PDF $p(t_2,s_2|t_1,s_1)$ for $s_2 > s_1$ is just

$$p(t_2,s_2|t_1,s_1) = \frac{1}{(s_2-s_1)^{1/\alpha}} L_\alpha\left(\frac{t_2-t_1}{(s_2-s_1)^{1/\alpha}}\right). \tag{27}$$

We remind the reader that $L_\alpha(x)=0$ for negative values of x. The expression for the joint PDF for multiple points is obvious.

IV. PROBABILITY DISTRIBUTION $h(s,t)$

The PDF's $h(s,t), h(s_2,t_2;s_1,t_1)$ of the inverse process $s = s(t)$ can be obtained from the PDF's of the process $t = t(s)$ with the help of relationship Eq. (16). We shall consider the single- and multiple-time cases separately. Again, due to the simple form of the Lévy distributions in Laplace space, we perform most of the calculations with Laplace transforms.

A. Single time

Using the notation $\tilde{h}(s,\lambda) = \mathcal{L}\{h(s,t)\}$ for the Laplace transform of $h(s,t)$ with respect to t, the relation Eq. (16) reads

$$\tilde{h}(s,\lambda) = -\frac{\partial}{\partial s}\left\langle \frac{1}{\lambda} e^{-\lambda t(s)} \right\rangle = -\frac{\partial}{\partial s}\frac{1}{\lambda} Z(s,\lambda). \tag{28}$$

The derivative with respect to s is easily performed with Eq. (21) and leads to the solution $\tilde{h}(s,\lambda)$:

$$\tilde{h}(s,\lambda) = \lambda^{\alpha-1} e^{-s\lambda^\alpha}. \tag{29}$$

This expression has already been derived in [7]—however, without giving a "simple physical argument." Here the derivation is clearly based on Eq. (14) which relates the Lévy-stable process and its inverse.

The inverse Laplace transform of Eq. (29) is known and has been calculated in [16]:

$$h(s,t) = \frac{1}{\alpha}\frac{t}{s^{1+1/\alpha}} L_\alpha\left(\frac{t}{s^{1/\alpha}}\right). \tag{30}$$

Moreover, in [17] the single-time distribution $h(s,t)$ has been identified as the Mittag-Leffler distribution:

$$h(s,t) = \sum_{n=0}^{\infty} \frac{(-st^\alpha)^n}{\Gamma(1+n\alpha)}. \tag{31}$$

Here we have obtained the PDF of $s(t)$ for single times. Therefore, a complete characterization of the inverse process is given in this case.

However, in order to derive an evolution equation for the PDF of the process $X(s(t))$ we require an equation that determines $h(s,t)$.

From Eq. (29) it is evident that $\tilde{h}(s,\lambda)$ obeys the differential equation

$$-\frac{\partial}{\partial s}\tilde{h}(s,\lambda) = \lambda^\alpha \tilde{h}(s,\lambda) \tag{32}$$

with the initial condition $\tilde{h}(0,\lambda) = \lambda^{\alpha-1}$ for $s=0$. Hence, Laplace inversion yields a fractional evolution equation for $h(s,t)$:

$$\frac{\partial}{\partial t} h(s,t) = -\,{}_0D_t^{1-\alpha}\frac{\partial}{\partial s} h(s,t). \tag{33}$$

The operator ${}_0D_t^{1-\alpha}$ denotes the Riemann-Liouville fractional differential operator, a possible generalization of integer order differentiation and integration to fractional orders (see

Appendix B). For a discussion of fractional derivatives we refer the reader to [18].

B. Multiple times

The statistical characterization of the process $s(t)$ for multiple times has been investigated from a mathematical point of view in the work of Bingham [17] already in 1971. He derived the following relationships for the moments $\langle s(t_N)\cdots s(t_1)\rangle$:

$$\frac{\partial^N}{\partial t_1 \ldots \partial t_N}\langle s(t_N)\cdots s(t_1)\rangle = \frac{1}{\Gamma(\alpha)^N}[t_1(t_2-t_1)\cdots(t_N-t_{N-1})]^{\alpha-1}. \tag{34}$$

This equation can be obtained from the previous relation (16), which implies the following relationship between the probability densities $p(t,s)$ and $h(s,t)$:

$$\frac{\partial}{\partial t}h(s,t) = -\frac{\partial}{\partial s}p(t,s),$$

$$\frac{\partial^2}{\partial t_1\,\partial t_2}h(s_2,t_2;s_1,t_2) = \frac{\partial^2}{\partial s_2\,\partial s_1}p(t_2,s_2;t_1,s_1),$$

$$\frac{\partial^N}{\partial t_1\cdots\partial t_N}h(s_N,t_N;\cdots;s_1,t_2) = (-1)^N\frac{\partial^N}{\partial s_N\cdots\partial s_1}p(t_N,s_N;\cdots;t_1,s_1). \tag{35}$$

In the following we shall derive explicit expressions for these moments and show that instead of (34) fractional equations can be used for their determination. Based on Eqs. (16) and (25) the derivation of an expression for the Laplace transform $\tilde{h}(s_2,\lambda_2;s_1,\lambda_1) := \mathcal{L}\{h(s_2,t_2;s_1,t_1)\}$ is obtained in a way analogous to the single-time case.

We start by considering Eq. (16) in Laplace space:

$$\tilde{h}(s_2,\lambda_2;s_1,\lambda_1) = \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\left\langle \frac{1}{\lambda_2}e^{-\lambda_2 t(s_2)}\frac{1}{\lambda_1}e^{-\lambda_1 t(s_1)}\right\rangle = \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\frac{1}{\lambda_1\lambda_2}Z(\lambda_2,s_2;\lambda_1,s_1). \tag{36}$$

Using Eq. (25) we can perform the derivatives of $Z(\lambda_2,s_2;\lambda_1,s_1)$ with respect to s_1,s_2:

$$\begin{aligned}\tilde{h}(s_2,\lambda_2;s_1,\lambda_1) = {}& \delta(s_2-s_1)\frac{\lambda_1^\alpha-(\lambda_1+\lambda_2)^\alpha+\lambda_2^\alpha}{\lambda_1\lambda_2}e^{-s_1(\lambda_1+\lambda_2)^\alpha} \\ &+\Theta(s_2-s_1)\frac{(\lambda_2^\alpha)[(\lambda_1+\lambda_2)^\alpha-\lambda_2^\alpha]}{\lambda_1\lambda_2} \\ &\times e^{-(\lambda_1+\lambda_2)^\alpha s_1}e^{-\lambda_2^\alpha(s_2-s_1)}+\Theta(s_1-s_2) \\ &\times\frac{(\lambda_1^\alpha)[(\lambda_1+\lambda_2)^\alpha-\lambda_1^\alpha]}{\lambda_1\lambda_2} \\ &\times e^{-(\lambda_1+\lambda_2)^\alpha s_2}e^{-\lambda_1^\alpha(s_1-s_2)}.\end{aligned} \tag{37}$$

As a result we have obtained the Laplace transform of the joint PDF $h(s_2,t_2;s_1,t_1)$. Unfortunately, a closed form of the inverse Laplace transform could not be calculated. The given solution $\tilde{h}$ can be readily used, however, to derive meaningful expressions that characterize the inverse process $s(t)$.

1. Moments of the inverse process

In order to obtain further information about the process $s(t)$ for multiple times we calculate the moments of the PDF. Let us first demonstrate how this can be achieved for the simple case $\langle s(t_1)s(t_2)\rangle$. This moment is defined from the PDF $h(s_2,t_2;s_1,t_1)$ as

$$\begin{aligned}\langle s(t_1)s(t_2)\rangle &= \int_0^\infty ds_1\int_0^\infty ds_2 s_1 s_2 h(s_2,t_2;s_1,t_1) \\ &= \mathcal{L}^{-1}\left\{\int_0^\infty ds_1\int_0^\infty ds_2 s_1 s_2 \tilde{h}(s_2,\lambda_2;s_1,\lambda_1)\right\},\end{aligned} \tag{38}$$

where the last step follows by interchanging inverse Laplace transform and integration. The integrations with respect to s_1,s_2 can be simply performed with the help of expression Eq. (36). The result is

$$\int_0^\infty ds_1\int_0^\infty ds_2 s_1 s_2\tilde{h}(s_2,\lambda_2;s_1,\lambda_1) = (\lambda_1+\lambda_2)^{-\alpha}\left\{\frac{\lambda_1^{-\alpha-1}}{\lambda_2}+\frac{\lambda_2^{-\alpha-1}}{\lambda_1}\right\}. \tag{39}$$

Now the inverse Laplace transform leads to an analytical solution for $\langle s(t_1)s(t_2)\rangle$ (see Appendix B):

$$\begin{aligned}\langle s(t_1)s(t_2)\rangle = {}& \Theta(t_2-t_1)\left\{\frac{1}{\Gamma(2\alpha+1)}t_1^{2\alpha}+\frac{1}{\Gamma(\alpha+1)^2}\right. \\ &\left.\times t_1^\alpha t_2^\alpha F\left(\alpha,-\alpha;\alpha+1;\frac{t_1}{t_2}\right)\right\}+\Theta(t_1-t_2) \\ &\times\left\{\frac{1}{\Gamma(2\alpha+1)}t_2^{2\alpha}+\frac{1}{\Gamma(\alpha+1)^2}t_1^\alpha t_2^\alpha\right. \\ &\left.\times F\left(\alpha,-\alpha;\alpha+1;\frac{t_2}{t_1}\right)\right\}.\end{aligned} \tag{40}$$

Here $F(a,b;c;z)$ denotes the hypergeometric function (see, e.g., Chap. 15 in [19]).

One notices that in the limit $t_2\to t_1$ expression (40) agrees with the second moment $\langle s(t)^2\rangle$:

$$\langle s(t)^2\rangle = \mathcal{L}^{-1}\left\{\int_0^\infty s^2\lambda^{\alpha-1}e^{-s\lambda^\alpha}ds\right\} = \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}, \tag{41}$$

where Eq. (29) has been used. The simple single time moment $\langle s(t)\rangle$ is given as $\langle s(t)\rangle = \mathcal{L}^{-1}\{\lambda^{-\alpha-1}\} = [1/\Gamma(\alpha+1)t^\alpha]$.

The calculation of higher order moments essentially follows the same steps.

Furthermore, we introduce the operator $(\partial/\partial t_1+\partial/\partial t_2)^{1-\alpha}$ in the sense of the single-time Riemann-Liouville fractional differential operator: $\mathcal{L}\{(\partial/\partial t_1+\partial/\partial t_2)^{-\alpha}g(t_1,t_2)\}=(\lambda_1+\lambda_2)^{-\alpha}\tilde{g}(\lambda_1,\lambda_2)$ (see Appendix A). An explicit expression in terms of an integral reads

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)^{-\alpha}g(t_1,t_2) = \frac{1}{\Gamma(\alpha)}\int_0^{\min(t_1,t_2)} dt' t'^{\alpha-1} g(t_1-t',t_2-t'). \qquad (42)$$

Using this fractional differential operator, we are in the position to write down a simple recursion relation for arbitrary moments of $h(\{s_i,t_i\})$. The second moment Eq. (39) reads

$$\langle s(t_1)s(t_2)\rangle = \left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)^{-\alpha}\{\langle s(t_1)\rangle + \langle s(t_2)\rangle\}. \qquad (43)$$

This immediately leads to (we assume $t_2>t_1$)

$$\langle s(t_2)s(t_1)\rangle = \left[{}_0D_{t_1}^{-\alpha}\{\langle s(t_2-\tilde{t}_1+t_1)\rangle + \langle s(t_1)\rangle\}\right]_{\tilde{t}_1=t_1}. \qquad (44)$$

The explicit expression allows one to obtain the fusion rule

$$\lim_{t_2\to t_1}\langle s(t_2)s(t_1)\rangle = \langle s(t_1)^2\rangle = 2\frac{1}{\Gamma(\alpha)}\int_0^{t_1} dt' t'^{\alpha-1}\langle s(t_1-t')\rangle = 2\,{}_0D_{t_1}^{-\alpha}s(t_1). \qquad (45)$$

The calculation of the third-order moment $\langle s(t_1)s(t_2)s(t_3)\rangle$ along the same lines yields the result

$$\langle s(t_1)s(t_2)s(t_3)\rangle = \left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_3}\right)^{-\alpha}\{\langle s(t_1)s(t_2)\rangle + \langle s(t_1)s(t_3)\rangle + \langle s(t_2)s(t_3)\rangle\}. \qquad (46)$$

The third moment is obtained via fractional integration of the sum of second-order moments. In the general case, the nth-order moment is calculated by fractional integration with respect to n times the sum of all permutations of $n-1$ order moments.

Due to the representation of the fractional operator

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_3}\right)^{-\alpha}g(t_1,t_2,t_3) = \frac{1}{\Gamma(\alpha)}\int_0^{\min(t_1,t_2,t_3)} dt' t'^{\alpha-1} g(t_1-t',t_2-t',t_3-t'), \qquad (47)$$

we can derive the fusion rule

$$\lim_{t_3\to t_1+0}\langle s(t_3)s(t_2)s(t_1)\rangle = \frac{1}{\Gamma(\alpha)}\int_0^{t_1} dt' t'^{\alpha-1}\{\langle s(t_1-t')s(t_1-t')\rangle + 2\langle s(t_2-t')s(t_1-t')\rangle\} = {}_0D_{t_1}^{-\alpha}\{\langle s(t_1)s(t_1)\rangle + 2\langle s(t_2-\tilde{t}_1+t_1)s(t_1)\rangle\}_{\tilde{t}_1=t_1}. \qquad (48)$$

The fusion $t_2\to t_1$ leads to

$$\langle s(t_1)^3\rangle = 3\,{}_0D_{t_1}^{-\alpha}\langle s(t_1)^2\rangle = 6D_{t_1}^{-\alpha}D_{t_1}^{-\alpha}\langle s(t_1)\rangle = 6\,{}_0D_{t_1}^{-2\alpha}\langle s(t_1)\rangle. \qquad (49)$$

The nth-order generalization reads

$$\langle s(t)^n\rangle = n!\,{}_0D_t^{-(n-1)\alpha}\langle s(t)\rangle. \qquad (50)$$

This equation can also be derived directly from $\tilde{h}(s,\lambda)$. Thus one can obtain a complete characterization of the process $s(t)$ based on Eq. (37) or Eq. (36), respectively. Below, we shall show how to obtain these results on the basis of an evolution equation for the multipoint PDF $h(s_1,t_1;\cdots;s_N,t_N)$.

2. The structure of the N-time PDF

From Eq. (16) one can derive the general form of the PDF h of the inverse process $s(t)$. The two-time PDF reads (here we assume the case $s_2>s_1$ for simplicity)

$$h(s_2,t_2;s_1,t_1) = \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\int_0^{t_1} dt_1'\int_0^{t_2} dt_2' p(t_2'-t_1',s_2-s_1)p(t_1',s_1) = -\frac{\partial}{\partial s_1}\int_0^{t_1} dt_1' h(s_2-s_1,t_2-t_1')p(t_1',s_1). \qquad (51)$$

We define

$$H(s_2-s_1,t_2-t_1;s_1-s_0,t_1-t_0) = -\frac{\partial}{\partial s_1}\int_0^{t_1} dt_1' h(s_2-s_1,t_2-t_1')p(t_1'-t_0,s_1-s_0). \qquad (52)$$

The form of the three-time PDF is obtained in the same way and reads for $s_3>s_2>s_1$

$$h(s_3,t_3;s_2,t_2;s_1,t_1) = \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\int_0^{t_1} dt_1'\int_0^{t_2} dt_2' h(s_3-s_2,t_3-t_2')p(t_2'-t_1',s_2-s_1)p(t_1',s_1) \qquad (53)$$

with a straightforward extension to the general case.

With the help of Eq. (52) this expression can be represented according to

$$h(s_3,t_3;s_2,t_2;s_1,t_1) = -\frac{\partial}{\partial s_1}\int_0^{t_1} dt_1' H(s_3-s_2,t_3-t_2;s_2-s_1,t_2-t_1')p(t_1',s_1). \qquad (54)$$

Recursively, we may define higher order functions

$$H^N(s_N-s_{N-1},t_N-t_{N-1};\cdots;t_1-t_0,s_1-s_0) = -\frac{\partial}{\partial s_1}\int_0^{t_1} dt_1' H^{N-1}(s_N-s_{N-1},t_N-t_{N-1};\cdots;s_2-s_1,t_2-t_1')p(t_1'-t_0,s_1,s_0). \qquad (55)$$

The integrals cannot simply be evaluated and the relations are formal. However, they show the underlying mathematical structure of the statistical description of the inverse process $s(t)$.

3. Fractional evolution equation

In analogy to the single-time case, where we have specified a fractional differential equation for $h(s,t)$, we now establish an evolution equation for $h(s_2,t_2;s_1,t_1)$.

From Eq. (37) it is evident that the following equation holds:

$$\left(\frac{\partial}{\partial s_1}+\frac{\partial}{\partial s_2}\right)\tilde{h}(s_2,\lambda_2;s_1,\lambda_1) = -(\lambda_1+\lambda_2)^\alpha \tilde{h}(s_2,\lambda_2;s_1,\lambda_1) \tag{56}$$

with initial conditions

$$\tilde{h}(0,\lambda_2;0,\lambda_1) = \frac{\lambda_1^\alpha - (\lambda_1+\lambda_2)^\alpha + \lambda_2^\alpha}{\lambda_1\lambda_2},$$

$$\tilde{h}(s_2,\lambda_2;0,\lambda_1) = \frac{(\lambda_2^\alpha)[(\lambda_1+\lambda_2)^\alpha - \lambda_2^\alpha]}{\lambda_1\lambda_2} e^{-\lambda_2^\alpha s_2},$$

$$\tilde{h}(0,\lambda_2;s_1,\lambda_1) = \frac{(\lambda_1^\alpha)[(\lambda_1+\lambda_2)^\alpha - \lambda_1^\alpha]}{\lambda_1\lambda_2} e^{-\lambda_1^\alpha s_1}. \tag{57}$$

A common way to solve first-order partial differential equations is the method of characteristics. Applying this method to Eq. (56) with the given initial condition for each case, one obtains the correct expressions Eq. (37). Therefore Eq. (56) determines the PDF in Laplace space.

Consequently, upon performing the inverse Laplace transform, we derive that $h(s_2,t_2;s_1,t_1)$ obeys the fractional evolution equation

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)h(s_2,t_2;s_1,t_1) = -\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)^{1-\alpha}\left(\frac{\partial}{\partial s_1} + \frac{\partial}{\partial s_2}\right)h(s_2,t_2;s_1,t_1), \tag{58}$$

where the fractional differential operator $(\partial/\partial t_1+\partial/\partial t_2)^{1-\alpha}$ has been defined according to $(\partial/\partial t_1+\partial/\partial t_2)^{1-\alpha}F(t_2,t_1) := (\partial/\partial t_1+\partial/\partial t_2)(\partial/\partial t_1+\partial/\partial t_2)^{-\alpha}F(t_2,t_1)$. The appearance of fractional time derivatives in Eq. (58) reveals the non-Markovian character of the stochastic process $s(t)$ and as a consequence of the coupled process $X(s(t))$.

The extension of the above result to n times is straightforward:

$$\left(\sum_{i=1}^N \frac{\partial}{\partial t_i}\right)h(\{s_i,t_i\}) = -\left(\sum_{i=1}^N \frac{\partial}{\partial t_i}\right)^{1-\alpha}\left(\sum_{i=1}^N \frac{\partial}{\partial s_i}\right)h(\{s_i,t_i\}). \tag{59}$$

Again we want to emphasize that this single evolution equation with the proper initial condition sufficiently describes the PDF for multiple times.

The above equation may also be used to calculate the moments $\langle s(t_N)\cdots s(t_1)\rangle$, which already have been specified above. The fractional evolution equation (59) implies the following relationship among the moments $\langle s(t_N)\cdots s(t_1)\rangle$:

$$\left(\sum_{i=1}^N \frac{\partial}{\partial t_i}\right)\langle s(t_N)\cdots s(t_1)\rangle = \left(\sum_{i=1}^N \frac{\partial}{\partial t_i}\right)^{1-\alpha}\{\langle s(t_{N-1})\cdots s(t_1)\rangle + \text{permutations}\}. \tag{60}$$

These equations are equivalent to the chain of equations (46) obtained by a direct inspection of the PDF's.

V. TWO-TIME MOMENTS OF THE DIFFUSION PROCESS

In this last section we focus on the usual diffusion process, i.e., we consider the Fokker-Planck operator

$$L = \frac{\partial^2}{\partial x^2}. \tag{61}$$

In this case, the moments are polynomials in s and we may directly use the results of the preceding section:

$$\langle x(s_2)x(s_1)\rangle = \Theta(s_2-s_1)s_1 + \Theta(s_1-s_2)s_2. \tag{62}$$

The corresponding moment with respect to time t is given by

$$\langle x(t_2)x(t_1)\rangle = \int_0^\infty\int_0^\infty ds_1 ds_2 h(s_2,t_2;s_1,t_1)\langle x(s_2)x(s_1)\rangle. \tag{63}$$

The integrations can be performed by inserting the PDF h in Laplace space:

$$\mathcal{L}\{\langle x(t_2)x(t_1)\rangle\} = \frac{(\lambda_1+\lambda_2)^\alpha}{\lambda_1\lambda_2}\int_0^\infty ds\, s\, e^{-(\lambda_1+\lambda_2)^\alpha s} = \frac{1}{(\lambda_1+\lambda_2)^\alpha \lambda_1\lambda_2}. \tag{64}$$

The inverse transform leads to the result

$$\langle x(t_2)x(t_1)\rangle = \frac{1}{\Gamma(\alpha+1)}\{\Theta(t_2-t_1)t_1^\alpha + \Theta(t_1-t_2)t_2^\alpha\} = \Theta(t_2-t_1)\langle s(t_1)\rangle + \Theta(t_1-t_2)\langle s(t_2)\rangle. \tag{65}$$

Similarly, we may calculate the moment $\langle x(t_2)^2 x(t_1)^2\rangle$:

$$\langle x(s_2)^2 x(s_1)^2\rangle = s_2 s_1 + 2\Theta(s_2-s_1)s_1^2 + 2\Theta(s_1-s_2)s_2^2. \tag{66}$$

This yields

$$\langle x(t_2)^2 x(t_1)^2\rangle = \langle s(t_2)s(t_1)\rangle + 2\Theta(t_2-t_1)\langle s(t_1)^2\rangle + 2\Theta(t_1-t_2)\langle s(t_2)^2\rangle. \tag{67}$$

For the evaluation of $\langle x(s_2)^{2m}x(s_1)^{2n}\rangle$ we may use the properties of the moments of Gaussian processes which read for $n > m$

$$\langle x(s_2)^{2m}x(s_1)^{2n}\rangle = A s_2^m s_1^n + B\Theta(s_2-s_1)s_1^{n-m}s_2^m + B\Theta(s_1-s_2)s_2^{n-m}s_1^m. \tag{68}$$

The coefficients A,B,C can be evaluated by an application of Wick's theorem for Gaussian processes.

The corresponding expression for the process $X(t)$ becomes accordingly

$$\langle x(t_2)^{2m}x(t_1)^{2n}\rangle = A\langle s(t_2)^m s(t_1)^n\rangle + B\Theta(t_2-t_1)\langle s(t_1)^{n-m}s(t_2)^m\rangle + B\Theta(t_1-t_2)\langle s(t_2)^{n-m}s(t_1)^m\rangle. \quad (69)$$

The calculation of the expectation values $\langle s(t_2)^{2m}s(t_1)^{2n}\rangle$ has been discussed above.

VI. CONCLUSION

Up to now the discussion of continuous time random walks and the corresponding fractional kinetic equations has been focused on single-time probability distributions only. On the basis of this PDF the scaling behavior of moments has been compared with experiments. However, more information has to be used in order to assign a definite stochastic process to a non-Markovian process. To this end we have considered multiple-time PDF for a certain class of stochastic processes.

Our approach is based on the framework of coupled Langevin equations (2) and (3) devised by Fogedby as a realization of a continuous time random walk. Here, the solution for the N-time PDF's are given as an integral transform of the PDF's of an accompanying Markovian process. We have shown that the non-Markovian character of this process can be traced back to the properties of the inverse Lévy-stable process.

The next step would be to compare these theoretical predictions with the behavior of physical systems which reveal subdiffusive behavior. To our knowledge multiple-time statistics of such systems have not yet been investigated experimentally. This would be of considerable interest. We may expect that in some cases the consideration of multiple-time statistics may lead to a more precise characterization of the underlying stochastic process.

It is well known that for the single-time case a fractional diffusion equation can be derived, which determines the PDF $f(x,t)$,

$$f(x,t) = \int_0^\infty ds\, h(s,t) f_1(x,s), \quad (70)$$

as a solution of

$$\frac{\partial}{\partial t} f(x,t) = {}_0D_t^{1-\alpha} L_{FP} f(x,t). \quad (71)$$

We would like to mention that a similar equation can be derived for the multiple-time PDF $f(x_2,t_2;x_1,t_1)$. This will be discussed in a future publication. The present article is a starting point for the investigation of multiple-time PDF's of the coupled Langevin equations of Fogedby.

ACKNOWLEDGMENTS

We gratefully acknowledge support by the Deutsche Forschungsgemeinschaft and wish to thank R. Hillerbrand, O. Kamps, and T. D. Frank for helpful discussions.

APPENDIX A: FRACTIONAL DIFFERENTIAL OPERATOR

The Riemann-Liouville fractional integral is defined as a generalization of the Cauchy formula to real orders α:

$${}_0D_t^{-\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(t')}{(t-t')^{1-\alpha}} dt' = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * g(t). \quad (A1)$$

Here $*$ denotes a Laplace convolution. Consequently, performing the Laplace transformation is straightforward and yields the well-known result

$$\mathcal{L}\{{}_0D_t^{-\alpha} g(t)\} = \lambda^{-\alpha}\tilde{g}(\lambda). \quad (A2)$$

From Eq. (A1) the Riemann-Liouville fractional differential operator is obtained by simple partial derivation:

$${}_0D_t^{1-\alpha} g(t) := \frac{\partial}{\partial t} {}_0D_t^{-\alpha} g(t). \quad (A3)$$

The extension of the fractional differential operator to two times t_1, t_2 is now obtained in a way analogous to the steps above.

First we define the fractional integral operator of two times in Laplace space:

$$\mathcal{L}\left\{ \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right)^{-\alpha} g(t_1,t_2) \right\} := (\lambda_1+\lambda_2)^{-\alpha} \tilde{g}(\lambda_1,\lambda_2). \quad (A4)$$

Furthermore the following equation holds:

$$\int_0^\infty dt_1 \int_0^\infty dt_2 e^{-\lambda_1 t_1 - \lambda_2 t_2} \frac{1}{\Gamma(\alpha)} t_1^{\alpha-1} \delta(t_2-t_1) = \int_0^\infty dt_1 e^{-t_1(\lambda_1+\lambda_2)} \frac{1}{\Gamma(\alpha)} t_1^{\alpha-1} = (\lambda_1+\lambda_2)^{-\alpha}. \quad (A5)$$

In physical time the fractional integral operator can thus be considered as an expression containing a twofold Laplace convolution with respect to t_1 and t_2, denoted with $**$:

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right)^{-\alpha} g(t_1,t_2) = \frac{1}{\Gamma(\alpha)} t_1^{\alpha-1} \delta(t_2-t_1) * * g(t_2,t_1) = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' t_1'^{\alpha-1} \times \delta(t_2',t_1') g(t_2-t_2', t_1-t_1'). \quad (A6)$$

Here we can distinguish between the cases $t_2 < t_1$ and $t_2 > t_1$ which results in Eq. (47). The fractional differential operator of two times is then, corresponding to Eq. (A3),

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right)^{1-\alpha} g(t_1,t_2) := \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right)^{-\alpha} g(t_1,t_2). \quad (A7)$$

In the general N-time case the fractional integral operator takes the form of an N-fold convolution

$$\left(\sum_{i=1}^N \frac{\partial}{\partial t_i} \right)^{-\alpha} g(t_1,\ldots,t_N) = \frac{1}{\Gamma(\alpha)} t_1^{\alpha-1} \delta(t_N - t_{N-1}) \cdots \delta(t_2 - t_1) * \cdots * g(t_1,\ldots,t_N), \quad (A8)$$

with Laplace transform

$$\mathcal{L}\left\{\left(\sum_{i=1}^{N}\frac{\partial}{\partial t_i}\right)^{-\alpha} g(t_1,\ldots,t_N)\right\} = \left(\sum_{i=1}^{N}\lambda_i\right)^{-\alpha}\tilde{g}(\lambda_1,\ldots,\lambda_N). \tag{A9}$$

APPENDIX B: CALCULATION OF MOMENTS

Using the results of the previous section we can explicitly write the second-order moment Eq. (43) as convolution integrals:

$$\langle s(t_1)s(t_2)\rangle = \frac{1}{\Gamma(\alpha)}\int_0^{t_1} dt_1'\int_0^{t_2} dt_2' t'^{\,\alpha-1}_1 \delta(t_2'-t_1')\left\{\frac{1}{\Gamma(\alpha+1)}(t_1 - t_1')^{\alpha} + \frac{1}{\Gamma(\alpha+1)}(t_2-t_2')^{\alpha}\right\}. \tag{B1}$$

If we distinguish between the cases $t_2 > t_1$ and $t_1 > t_2$ in order to perform the integrations, we obtain

$$\begin{aligned}\langle s(t_1)s(t_2)\rangle = {} & \Theta(t_2-t_1)\left\{\frac{1}{\Gamma(2\alpha+1)}t_1^{2\alpha} + \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^{t_1} dt'\, t'^{\,\alpha-1}(t_2-t')^{\alpha}\right\} \\ & + \Theta(t_1-t_2)\left\{\frac{1}{\Gamma(2\alpha+1)}t_2^{2\alpha} + \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^{t_2} dt'\, t'^{\,\alpha-1}(t_1-t')^{\alpha}\right\}.\end{aligned} \tag{B2}$$

The integrals can be performed with MAPLE and lead to the hypergeometric function $F(a,b;c;z)$:

$$\int_0^{t_1} dt'\, t'^{\,\alpha-1}(t_2-t')^{\alpha} = \frac{1}{\alpha}t_1^{\alpha}t_2^{\alpha}F\left(\alpha,-\alpha;\alpha+1;\frac{t_1}{t_2}\right). \tag{B3}$$

[1] E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).
[2] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[3] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[4] R. Friedrich, Phys. Rev. Lett. **90**, 084501 (2003).
[5] A. M. Obukhov, Adv. Geophys. **6**, 113 (1959).
[6] R. Metzler and J. Klafter, J. Phys. A **37**, R161 (2004).
[7] H. C. Fogedby, Phys. Rev. E **50**, 1657 (1994).
[8] M. M. Meerschaert, D. A. Benson, and B. Baeumer, Phys. Rev. E **63**, 021112 (2001).
[9] M. M. Meerschaert and H. P. Scheffler, J. Appl. Probab. **41**, 623 (2004).
[10] P. Becker-Kern, M. M. Meerschaert, and H. P. Scheffler, Ann. Prob. **32**, 730 (2004).
[11] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1998).
[12] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 2004).
[13] D. Schertzer, M. Larcheveque, J. Duan, V. V. Yanovsky, and S. Lovejoy, J. Math. Phys. **42**, 1 (2001).
[14] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley Reading, MA, 1954).
[15] V. V. Yanovsky, A. V. Chechkin, D. Schertzer, and A. V. Tur, Physica A **282**, 293 (1999).
[16] E. Barkai, Phys. Rev. E **63**, 046118 (2001).
[17] N. H. Bingham, Z. Wahrscheinlichkeitstheor. Verwandte Geb. **17**, 1 (1971).
[18] I. Podlubny, *Fractional Differential Equations* (Academic, New York, 1999).
[19] *Handbook of Mathematical Functions*, edited by M. Abramowitz and C. A. Stegun (Dover, New York, 1972).